A SIMPLE DERIVATION OF BV BOUNDS FOR INHOMOGENEOUS RELAXATION SYSTEMS [*]

BENOÎT PERTHAME [†], NICOLAS SEGUIN [‡], AND MAGALI TOURNUS [§]

Abstract. We consider relaxation systems of transport equations with heterogeneous source terms and with boundary conditions, which limits are scalar conservation laws. Classical bounds fail in this context and in particular BV estimates. They are the most standard and simplest way to prove compactness and convergence.

We provide a novel and simple method to obtain partial BV regularity and strong compactness in this framework. The standard notion of entropy is not convenient either and we also indicate another, but closely related, notion. We give two examples motivated by renal flows which consist of 2 by 2 and 3 by 3 relaxation systems with 2-velocities but the method is more general.

Key words. Hyperbolic relaxation; spatial heterogeneity; entropy condition; boundary conditions; strong compactness.

Subject classifications. 35L03, 35L60, 35B40, 35Q92

1. Introduction

The usual framework of hyperbolic relaxation [4, 16, 3] concerns the convergence of a general hyperbolic system with stiff source terms toward a conservation law, when the relaxation parameter ϵ goes to zero. More specifically, Jin and Xin [12] introduced a 2×2 linear hyperbolic system with stiff source term that approximates any given conservation law. The problem of interest is to prove the convergence of the microscopic quantities depending on ϵ toward the macroscopic quantities. A problem entering the framework of hyperbolic relaxation is motivated by very simplified models of kidney physiology [21, 22, 20] and fits the Jin and Xin framework with two major differences, boundary conditions, and spatial dependence of the source term. The type of boundary condition and the spatial dependence constitute the main novelty of the present study.

The system represents two solute concentrations $u_\epsilon(x,t)$ and $v_\epsilon(x,t)$ and is written, for $t \geq 0$ and $x \in [0,L]$,

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(x,t) + \dfrac{\partial u_\epsilon}{\partial x}(x,t) = \dfrac{1}{\epsilon}\Big[h(v_\epsilon(x,t),x)) - u_\epsilon(x,t)\Big], \\[2mm] \dfrac{\partial v_\epsilon}{\partial t}(x,t) - \dfrac{\partial v_\epsilon}{\partial x}(x,t) = \dfrac{1}{\epsilon}\Big[u_\epsilon(x,t) - h(v_\epsilon(x,t),x))\Big], \\[2mm] u_\epsilon(0,t) = u_0, \qquad v_\epsilon(L,t) = \alpha u_\epsilon(L,t), \qquad \alpha \in (0,1), \\[2mm] u_\epsilon(x,0) = u^0(x) > 0, \qquad v_\epsilon(x,0) = v^0(x) > 0. \end{cases} \qquad (1.1)$$

The question of interest here is to understand the behavior of u_ϵ and v_ϵ when ϵ vanishes. Another question, addressed in [19], is to explain the urine concentration mechanism due to the 'pumps', on the cell membranes, represented by the nonlinearity $h(v,x)$.

We make the following hypotheses

$$h(0,x) = 0, \qquad 1 < \beta \leq \frac{\partial h}{\partial v}(v,x) \leq \mu, \tag{1.2}$$

$$\sup_v \int_0^L |\frac{\partial h}{\partial x}(v,x)| dx \leq C, \qquad h(.,x) \text{ is not locally affine.} \tag{1.3}$$

Here C, β and μ are positive constants. Also, we use the following bounds on the initial data

$$u^0, v^0 \in L^\infty(0,L), \qquad \frac{d}{dx} u^0 \in L^1(0,L), \qquad \frac{d}{dx} v^0 \in L^1(0,L), \tag{1.4}$$

and, the well-prepared initial data condition

$$u^0(x) = h(v^0(x),x), \qquad x \in [0,L]. \tag{1.5}$$

It is simple and standard to find the formal limit as ϵ vanishes. Adding the two equations of (1.1), we find the conservation law

$$\frac{\partial [u_\epsilon + v_\epsilon](x,t)}{\partial t} + \frac{\partial [u_\epsilon - v_\epsilon](x,t)}{\partial x} = 0. \tag{1.6}$$

We expect that, in the limit, the idendity holds

$$u(x,t) = h(v(x,t),x)$$

and this identifies the limiting quasi-linear conservation law

$$\frac{\partial [h(v(x,t),x) + v(x,t)]}{\partial t} + \frac{\partial [h(v(x,t),x) - v(x,t)]}{\partial x} = 0, \tag{1.7}$$

and the boundary conditions on $\rho(x,t) = h(v(x,t),x) + v(x,t)$, namely $\rho(0,t) = u_0 + h^{-1}(u_0,0)$ and $\rho(L,t)$ is free, can be identified via the standard argument of Bardos-Le Roux and Nédélec, [2], using the adapted Kružkov entropies that we introduce in Section 3.

The justification poses particular difficulties due to the boundary condition and x-dependent flux.

One difficulty is that, in order to justify the above limit, an entropy identity is needed for two reasons. Firstly to prove that equilibrium is reached [23]. Secondly, because at the limit $\epsilon = 0$, a quasi-linear equation arises, shocks can be produced. Therefore, an entropy formulation is needed to define uniquely the solutions. When applied to this conservation equation, usual convex entropies as both in [6] or Kružkov [14], contain a term with a derivative of h with respect to x. When the regularity of h is limited to BV, h_x is a measure and the weak entropy formulation is not well defined. One of our goal is to present a convenient notion of entropy with x-dependent relaxation in order to get rid of this problem. We use an entropy formulation introduced in [1] and adapted to scalar conservation laws with spatial heterogeneities.

Another difficulty is that, in order to prove the validity of the limit, strong compactness is also needed. Several methods have been developed in this goal. Uniform bounds in the space of functions with bounded variation (BV in short) is the strongest method and the most standard. The homogeneous version of the system (1.1), on $\mathbb{R}^n$ is treated in [13], where an L^1 contraction property is used to establish BV bounds in x.

A similar approach is used in [17] where the authors deal with a similar homogeneous system with boundary conditions; the result is extended to a system with a source term in [24]. The authors of [10] also derive BV estimates for an inhomogeneous system that relaxes to a homogeneous system. More recently, and for the purpose of internal layers, the authors of [5] are interested in a similar setting and also study the limit when the relaxation time goes to zero only on a sub-domain of the space.. More elaborate tools are the compensated compactness method [15] or the kinetic formulation and averaging lemmas [9, 18] (in particular extending the measure valued solutions of DiPerna [8]). These methods are weaker, because they give only convergence but no bounds, and thus apply to more general situations than BV bounds. Because of the spatial x-dependence of h, the BV framework is not available as today for (1.1). Whereas time BV estimates follow immediately from the equations, compactness in the spatial direction cannot be obtained in this way. We propose a new method to prove spatial compactness. It does not use BV bounds on each component, but gives BV bounds in x for a single quantity and can be applied to spatially heterogeneous systems. We are going to prove that

PROPOSITION 1.1. *We make assumptions* (1.2), (1.3), (1.4), (1.5) *and fix a time T. We have*
(i) u_ϵ and v_ϵ are uniformly bounded in $L^\infty([0,L] \times [0,T])$,
(ii) $\frac{\partial}{\partial t} u_\epsilon$, $\frac{\partial}{\partial t} v_\epsilon$ and $\frac{\partial}{\partial x}[u_\epsilon - v_\epsilon]$ are bounded in $L^\infty([0,T]; L^1(0,L))$,
(iii) there exists $v \in L^\infty([0,L] \times [0,T])$ such that

$$u_\epsilon(x,t) \underset{\epsilon \to 0}{\longrightarrow} h(v(x,t),x), \qquad v_\epsilon(x,t) \underset{\epsilon \to 0}{\longrightarrow} v(x,t), \, a.e. \qquad (1.8)$$

(iv) the equation (1.7) *is satisfied and entropy inequalities hold, see section 3.*

However it is not correct that $\frac{\partial}{\partial x} u_\epsilon$ and $\frac{\partial}{\partial x} v_\epsilon$ are separately bounded in $L^\infty([0,T]; L^1(0,L))$.

2. L^∞ bound

We first prove uniform estimates. Unlike the homogeneous case, L^∞ bounds are not always true and the most general existence theory is in L^1, see [7]. Here we build particular sub and supersolutions of (1.1) which are uniformly bounded in ϵ.

LEMMA 2.1. *The solution of* (1.1) *satisfies the uniform estimate*

$$\|u_\epsilon\|_{L^\infty([0,L] \times [0,T])} \leq K(\beta, u_0, u^0, v^0), \qquad \|v_\epsilon\|_{L^\infty([0,L] \times [0,T])} \leq K(\beta, u_0, u^0, v^0). \quad (2.1)$$

Proof. To obtain an L^∞ bound on the time-dependent solution, we follow the approach of [1] and use the comparison principle with appropriate supersolution. Indeed, because of the $x-$dependence of h, constant functions are not super-solution of the stationary problem. We introduce the stationary version of (1.1)

$$\begin{cases} \dfrac{dU_\epsilon}{dx}(x) = \dfrac{1}{\epsilon}\Big[h(V_\epsilon(x),x)) - U_\epsilon(x)\Big], \\[2mm] -\dfrac{dV_\epsilon}{dx}(x) = \dfrac{1}{\epsilon}\Big[U_\epsilon(x) - h(V_\epsilon(x),x))\Big], \\[2mm] \quad U_\epsilon(0) = U_0 > 0, \qquad V_\epsilon(L) = \alpha U_\epsilon(L). \end{cases} \tag{2.2}$$

We are going to prove that there exists a smooth super solution (U_ϵ, V_ϵ) of the stationary problem (2.2), and a constant $K(U_0, \beta) > 0$ such that

$$0 \le U_\epsilon(x) \le K(U_0, \beta), \qquad 0 \le V_\epsilon \le K(U_0, \beta). \tag{2.3}$$

This concludes the proof of Lemma 2.1 because a solution of (2.2) where $U_0 \ge u_0$, is a super-solution of (1.1), and the comparison principle gives $0 \le u_\epsilon \le U_\epsilon$ and $0 \le v_\epsilon \le V_\epsilon$.

Because it has been proved in [21] that (2.2) admits a unique solution which lies in $\mathrm{BV}([0,L] \times [0,T])$ (fixed point argument for the existence and contraction for the uniqueness), it remains to prove that a solution of (2.2) with $U_0 \ge u_0$ is uniformly bounded in ϵ.

Adding the two lines of (2.2), we obtain a quantity which does not depend on x,

$$U_\epsilon(x) - V_\epsilon(x) =: K_\epsilon. \tag{2.4}$$

Using the boundary values, we find uniform bounds on K_ϵ

$$K_\epsilon = U_0 - V_\epsilon(0) \le U_0 \qquad K_\epsilon = U_\epsilon(L) - V_\epsilon(L) = (1-\alpha)U_\epsilon(L) \ge 0, \tag{2.5}$$

and thus

$$0 \le K_\epsilon \le U_0, \qquad U_\epsilon(L) \le \dfrac{U_0}{1-\alpha}. \tag{2.6}$$

Hence, we just have to prove that U_ϵ is uniformly bounded in L^∞, knowing that U_0 and $U_\epsilon(L)$ are uniformly bounded in $\mathbb{R}$. For that, we use the maximum principle assuming $\mathcal{C}^1$ regularity (one can easily justify it for a regularized function $h(v,x)$ and pass to the limit). Indeed, if U_ϵ reaches its maximal value on the boundary, the result follows from (2.6). If U_ϵ reaches its maximal value at $x_0 \in]0, L[$, then,

$$0 = \dfrac{dU_\epsilon}{dx}(x_0) = \dfrac{1}{\epsilon}\Big(h(U_\epsilon(x_0) - K_\epsilon, x_0) - U_\epsilon(x_0)\Big), \tag{2.7}$$

and thus by assumption (1.2),

$$U_\epsilon(x_0) = h(U_\epsilon(x_0) - K_\epsilon, x_0) \ge \beta U_\epsilon(x_0) - \beta K_\epsilon. \tag{2.8}$$

Finally, (2.3) is proved because the above inequality gives

$$U_\epsilon(x_0) \le \dfrac{\beta}{\beta-1} K_\epsilon. \tag{2.9}$$

□

3. Adapted (heterogeneous) entropies

As explained in the introduction, entropies are useful to derive additional bounds and to characterize the limit as ϵ vanishes, see [11, 4, 23]. We are going to use specific entropies adapted to spatial dependence.

We recall the usual approach which is to define

DEFINITION 3.1 (Entropy pair). *We call an entropy pair for the system* (1.1) *a couple of functions* (S, Σ) *in* $BV\big([0,1], \mathcal{C}(\mathbb{R})\big)$ *which satisfy*

$$(i)\, S(.,x) \quad and \quad \Sigma(.,x)\, are\, convex, \qquad (ii)\, \frac{\partial S}{\partial u}(h(v,x),x) = \frac{\partial \Sigma}{\partial v}(v,x). \qquad (3.1)$$

For such entropy pairs, it is immediate to check that

$$\frac{\partial}{\partial t}\big[S(u_\epsilon,x) + \Sigma(v_\epsilon,x)\big] + \frac{\partial}{\partial x}\big[S(u_\epsilon,x) - \Sigma(v_\epsilon,x)\big]$$

$$= \frac{1}{\epsilon}\Big(\frac{\partial S}{\partial v}(u_\epsilon,x) - \frac{\partial \Sigma}{\partial v}(v_\epsilon,x)\Big)\Big(h(v_\epsilon,x) - u_\epsilon\Big) + \frac{\partial S}{\partial x}(u_\epsilon,x) - \frac{\partial \Sigma}{\partial x}(v_\epsilon,x).$$

Now, using (3.1)(ii), we can write

$$\frac{\partial}{\partial t}\big[S(u_\epsilon,x) + \Sigma(v_\epsilon,x)\big] + \frac{\partial}{\partial x}\big[S(u_\epsilon,x) - \Sigma(v_\epsilon,x)\big]$$

$$= \frac{1}{\epsilon}\Big(\frac{\partial S}{\partial v}(u_\epsilon,x) - \frac{\partial S}{\partial v}(h(v_\varepsilon,x),x)\Big)\Big(h(v_\epsilon,x) - u_\epsilon\Big) + \frac{\partial S}{\partial x}(u_\epsilon,x) - \frac{\partial \Sigma}{\partial x}(v_\epsilon,x)$$

$$\leq \frac{\partial S}{\partial x}(u_\epsilon,x) - \frac{\partial \Sigma}{\partial x}(v_\epsilon,x),$$

because S is convex with respect to its first variable and thus $\dfrac{\partial S}{\partial v}$ is non decreasing with respect to its first variable.

The shortcoming of Definition 3.1 is that the above right hand side is not always well defined for u_ϵ and v_ϵ BV functions. Indeed, being given $S(u,x)$, we compute

$$\Sigma(v,x) = \int_0^v S'_u\big(h(\bar{v},x),x\big)d\bar{v}$$

and the expression for the x-derivative

$$\frac{\partial \Sigma}{\partial x}(v,x) = \int_0^v \big[S''_{uu}\big(h(\bar{v},x),x\big)h_x(\bar{v},x) + \cdots\big]d\bar{v}$$

does not make intrinsic sense.

However, this entropy inequality is enough to prove that equilibrium is reached. Because the expressions $\dfrac{\partial S}{\partial x}(u_\epsilon,x)$ and $\dfrac{\partial \Sigma}{\partial x}(v_\epsilon,x)$ are bounded thanks to our assumptions on h, we may choose

$$S(u) = \frac{u^2}{2}, \qquad \Sigma(v,x) = \int_0^v h(\bar{v},x)d\bar{v},$$

and using the entropy dissipation, with assumption (1.3), we find after integration of equality (3) in (x,t), that for all $T > 0$ for some constant $C(T)$ which does not depend on ε, it holds

$$\frac{1}{\epsilon}\int_0^T \int_0^L \big|u_\epsilon(x,t) - h(v_\epsilon(x,t),x)\big|^2 dxdt \leq C(T).$$

Therefore, we arrive at the conclusion

PROPOSITION 3.2. *For u_ϵ and v_ϵ solutions of* (1.1)*, we have the convergence*

$$u_\epsilon - h(v_\epsilon, x) \underset{\epsilon \to 0}{\longrightarrow} 0, \qquad L^2\Big([0,L] \times [0,T]\Big). \tag{3.2}$$

This convergence result toward equilibrium is an intermediate step in the proof of Proposition 1.1.

We wish to go further and avoid the two terms containing x-derivatives. For this goal, we define adapted (heterogeneous) entropies by imposing more restrictions on the spatial dependence of the entropy pair.

DEFINITION 3.3 (An adapted (heterogeneous) entropy family). *The pair of continuous functions (S, Σ), is an adapted (heterogeneous) entropy for the system* (1.1) *if it satisfies the conditions of Definition 3.1 and if*

$$-\frac{\partial S}{\partial x}(h(v,x),x) + \frac{\partial \Sigma}{\partial x}(v,x) = 0, \qquad \forall v \geq 0. \tag{3.3}$$

An example of such a family of entropies parametrized by $p \in \mathbb{R}$ are the adapted Kružkov entropies

$$S_p(u,x) = |u - h(k_p(x),x)|, \qquad \Sigma_p(v,x) = |v - k_p(x)|, \tag{3.4}$$

where $(k_p)_{p \in \mathbb{R}}$ is the family of stationary solutions of the limit equation (1.7), which is equivalent to say

$$h(k_p(x),x) - k_p(x) = p.$$

With this choice of an entropy pair, the above entropy inequality then reduces to

$$\frac{\partial}{\partial t}[S_p(u_\epsilon,x) + \Sigma_p(v_\epsilon,x)] + \frac{\partial}{\partial x}[S_p(u_\epsilon,x) - \Sigma_p(v_\epsilon,x)] \leq 0.$$

As a consequence, thanks to the strong compactness proven in next part, in the limit $\varepsilon \to 0$, the quasilinear conservation law (1.7) comes with the family of adapted Kružkov entropies. Indeed, if we define

$$\rho(x,t) := h(v(x,t),x) + v(x,t), \qquad A(\rho,t) := h(v(x,t),x) - v(x,t), \tag{3.5}$$

then, as in [1] and because h is increasing, the following entropy inequality holds in the sense of distributions

$$\frac{\partial}{\partial t}\Big|\rho(x,t) - \Big(k_p(x) + h(k_p(x),x)\Big)\Big| + \frac{\partial}{\partial x}\Big|A(\rho,x) - A\Big(k_p(x) + h(k_p(x),x),x\Big)\Big| \leq 0. \tag{3.6}$$

4. BV bounds

For well prepared initial conditions (1.5), we present here our method to prove BV bounds for appropriate quantities and strong compactness for u_ϵ and v_ϵ, that are points (ii), (iii) of Proposition 1.1.

1st step. A bound on the time derivative at $t=0$. Our first statement is

$$\int_0^L |\frac{\partial u_\epsilon}{\partial t}(x,0)|dx \leq K_1(u^0), \qquad \int_0^L |\frac{\partial v_\epsilon}{\partial t}(x,0)|dx \leq K_2(v^0). \tag{4.1}$$

Indeed, because initial conditions are at equilibrium, we have $\frac{\partial v_\epsilon}{\partial t}(x,0) - \frac{\partial v_\epsilon}{\partial x}(x,0) = 0$.

We multiply this equality by $\text{sign}\left(\frac{\partial}{\partial t} v_\epsilon\right)(x,0)$ and integrate over $[0,L]$ to get

$$\int_0^L \left|\frac{\partial v_\epsilon}{\partial t}(x,0)\right| dx = \int_0^L \left|\frac{\partial v_\epsilon}{\partial x}(x,0)\right| dx \le K_2(v^0), \tag{4.2}$$

and the above inequality follows from assumption (1.4). This gives the first inequality of estimates (4.1). The same argument applies for u_ϵ.

2nd step. The time BV estimate. To prove (2.1), we differentiate each line of (1.1) with respect to time and we multiply it respectively by $\text{sign}\left(\frac{\partial}{\partial t} u_\epsilon\right)$ and $\text{sign}\left(\frac{\partial}{\partial t} v_\epsilon\right)$ and integrate in x. Adding the two lines, we obtain

$$\frac{d}{dt}\int_0^L [|\frac{\partial}{\partial t} u_\epsilon| + |\frac{\partial}{\partial t} v_\epsilon|](x,t)dx \le |\frac{\partial}{\partial t} u_0| - |\frac{\partial}{\partial t} u_\epsilon(L,t)| + |\frac{\partial}{\partial t} v_\epsilon(L,t)| - |\frac{\partial}{\partial t} v_\epsilon(0,t)|$$

$$= -\frac{1}{2}|\frac{\partial}{\partial t} u_\epsilon(L,t)| - |\frac{\partial}{\partial t} v_\epsilon(0,t)| \le 0, \tag{4.3}$$

which implies, using estimate (4.1),

$$\int_0^L [|\frac{\partial}{\partial t} u_\epsilon| + |\frac{\partial}{\partial t} v_\epsilon|](x,t)dx \le \int_0^L [|\frac{\partial}{\partial t} u_\epsilon| + |\frac{\partial}{\partial t} v_\epsilon|](x,0)dx \le K_1(u^0) + K_2(v^0). \tag{4.4}$$

3rd step. BV bound in x. We complete the proof Proposition 1.1 (ii). Because of space dependence of h we cannot apply the same arguments for x-derivatives and build a single BV quantity. We add the two lines of (1.1) and obtain

$$\left(\frac{\partial}{\partial x}(u_\epsilon - v_\epsilon)\right)(x,t) = -\left(\frac{\partial}{\partial t}(u_\epsilon + v_\epsilon)\right)(x,t).$$

Using (2.1), we thus conclude that for all $t \ge 0$

$$\int_0^L \left|\frac{\partial}{\partial x}(u_\epsilon - v_\epsilon)\right|(x,t)dx \le K_1(u^0) + K_2(v^0).$$

4th step. Compactness.

Therefore we can conclude that $\left(u_\epsilon - v_\epsilon\right)$ is compact in $L^1\left([0,L] \times [0,T]\right)$. Now, thanks to Proposition (3.2), $\left(h(v_\epsilon,.) - u_\epsilon\right)$ is compact in $L^1\left([0,L] \times [0,T]\right)$. A combination of these two last compact embeddings gives us that

$$h(v_\epsilon,.) - v_\epsilon \qquad \text{is compact in } L^1\left([0,L] \times [0,T]\right). \tag{4.5}$$

Therefore, there is a function $Q \in L^\infty(0,L)$ such that, after extraction of a subsequence,

$$h(v_\epsilon,.) - v_\epsilon \underset{\epsilon \longrightarrow 0}{\longrightarrow} Q(x),$$

and because $v \mapsto h(v,x) - v$ is one-to-one, thanks to assumptions (1.2). In the same way, we conclude that the sequence v_ϵ converges. Gathering the informations above, Proposition 1.1 (iii) is proved.

Then we can pass to the limit and obtain the last statement, Proposition 1.1 (iv).

5. Extension to a more specific relaxation system

The method we have developed so far can be extended to a more realistic problem arising in kidney physiology that motivated this study. The system introduced and studied in [21] is written, for $t \geq 0$ and $x \in [0, L]$,

$$
\begin{cases}
\dfrac{\partial C_\epsilon^1}{\partial t}(x,t) + \dfrac{\partial C_\epsilon^1}{\partial x}(x,t) = \dfrac{1}{3\epsilon}\Big[C_\epsilon^2(x,t) + h(C_\epsilon^3(x,t),x)) - 2C_\epsilon^1(x,t)\Big], \\[2mm]
\dfrac{\partial C_\epsilon^2}{\partial t}(x,t) + \dfrac{\partial C_\epsilon^2}{\partial x}(x,t) = \dfrac{1}{3\epsilon}\Big[C_\epsilon^1(x,t) + h(C_\epsilon^3(x,t),x)) - 2C_\epsilon^2(x,t)\Big], \\[2mm]
\dfrac{\partial C_\epsilon^3}{\partial t}(x,t) - \dfrac{\partial C_\epsilon^3}{\partial x}(x,t) = \dfrac{1}{3\epsilon}\Big[C_\epsilon^1(x,t) + C_\epsilon^2(x,t) - 2h(C_\epsilon^3(x,t),x))\Big], \\[2mm]
C_\epsilon^1(0,t) = C_0^1, \qquad C_\epsilon^2(0,t) = C_0^2, \qquad C_\epsilon^3(L,t) = C_\epsilon^2(L,t), \qquad t > 0,
\end{cases}
\tag{5.1}
$$

Again, we want to prove uniform BV bounds for a small parameter ϵ, which measures the ratio between ionic exchanges and flow along the tubules.

We make the same assumptions (1.2) (the condition $1 < \beta \leq \dfrac{\partial h}{\partial v}(v,x)$ can be relaxed to $1 \leq \dfrac{\partial h}{\partial v}(v,x)$) and (1.3) and same hypotheses on the initial conditions, namely they belong to BV and are at equilibrium which means

$$C^1 = C^2 = h(C^3, x).$$

Following (3.5), the conservative quantity ρ and the flux B are defined by

$$\rho(x,t) := 2h(C^3(x,t),x) + C^3(x,t), \quad B(\rho,t) := 2h(C^3(x,t),x) - C^3(x,t). \tag{5.2}$$

For $p \in \mathbb{R}$, we define uniquely the steady state k_p as

$$B(k_p(x),x) = p. \tag{5.3}$$

THEOREM 5.1 (Limit $\varepsilon \to 0$). *The functions C_ϵ^i, $i = 1, 2, 3$ converge almost everywhere to bounded functions C^i and the quantity $\rho(x,t)$ is an entropy solution to*

$$
\begin{cases}
\dfrac{\partial}{\partial t}\rho(x,t) + \dfrac{\partial}{\partial x}B(\rho(x,t),x) = 0, \qquad t > 0,\, x \in [0, L], \\[3mm]
\rho(0,t) = C_0^1 + C_0^2 + h^{-1}\Big(\dfrac{C_0^1 + C_0^2}{2}, 0\Big), \qquad t > 0, \\[3mm]
\rho(x,0) = \rho^0(x), \qquad \rho^0(x) := C^0(x) + 2h(C^0(x),x), \qquad x \in [0, L].
\end{cases}
\tag{5.4}
$$

The entropy formulation of the conservation law, for adapted (heterogeneous) entropies is written, following [1] again,

$$\dfrac{\partial}{\partial t}\Big|\rho(x,t) - \Big(2k_p(x) + h(k_p(x),x)\Big)\Big| + \dfrac{\partial}{\partial x}\Big|B(\rho,x) - B\Big(2k_p(x) + h(k_p(x),x),x\Big)\Big| \leq 0.$$

This family of inequalities is enough to prove uniqueness as in [17] (see [20] for details).

The boundary conditions are understood in the following sense

Boundary condition at $x = 0$.   For all k_p such that $k_p(0) + 2h(k_p(0),0) \in$

$I\Big(\rho(0,t),h^{-1}\big(\frac{C_0^1+C_0^2}{2},0\big)+C_0^1+C_0^2\Big)$, we have

$$\text{sign}\Big(\rho(0,t)-h^{-1}(\frac{C_0^1+C_0^2}{2},0)-C_0^1-C_0^2\Big) \\ \Big(B(\rho(0,t),0)-[2h(k_p(0),0)-k_p(0)]\Big)\leq 0, \tag{5.5}$$

Boundary condition at $x=L$. For all k_p such that $k_p(L)+2h(k_p(L),L)\in I\Big(\rho(L,t),2w_L+h^{-1}(w_L,L)\Big)$, we have

$$\text{sign}\Big(\rho(L,t)-2w_L-h^{-1}(w_L,L)\Big)\Big(B(\rho(L,t),L)-[2h(k_p(L),L)-k_p(L)]\Big)\geq 0, \quad (5.6)$$

where $w_L(t):=\lim_{\varepsilon\longrightarrow 0}C_\varepsilon^1(L,t)$, and where $I(a,b)$ denotes the interval $(\min(a,b),\max(a,b))$.

Following the arguments we gave for the 2×2 system, and that we do not repeat, we can prove BV bounds in several steps
(i) C_ϵ^1, C_ϵ^2 and C_ϵ^3 are bounded in $L^\infty((0,\infty)\times(0,L))$,
(ii) $C_\epsilon^2(x,t)+h(C_\epsilon^3(x,t),x))-2C_\epsilon^1(x,t)\xrightarrow[\epsilon\to 0]{}0$, $\;C_\epsilon^1(x,t)+h(C_\epsilon^3(x,t),x))-2C_\epsilon^2(x,t)\xrightarrow[\epsilon\to 0]{}0$ in $L^2((0,\infty)\times(0,L))$,
(iii) $\frac{\partial C_\epsilon^1}{\partial t},\frac{\partial C_\epsilon^2}{\partial t},\frac{\partial C_\epsilon^3}{\partial t}$ are bounded in $L^\infty\big((0,\infty;L^1(0,L)\big)$,
(iv) $\frac{\partial C_\epsilon^1}{\partial x}+\frac{\partial C_\epsilon^2}{\partial x}+\frac{\partial C_\epsilon^3}{\partial x}$ is bounded in $L^\infty\big((0,\infty;L^1(0,L)\big)$.
These statements prove the convergence result in Theorem 5.1.

REFERENCES

[1] E. Audusse and B. Perthame, *Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies*, Proc. Roy. Soc. Edinburgh Sect. A, 135 (2005), pp. 253–265.
[2] C. Bardos, A. Y. le Roux, and J.-C. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, 4 (1979), pp. 1017–1034.
[3] F. Bouchut, *Non linear stability of finite volume methods for hyperbolic conservation laws and well balanced schemes for sources*, Birkhaüser-Verlag, 2004.
[4] G. Q. Chen, C. D. Levermore, and T. P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math., 47 (1994), pp. 787–830.
[5] F. Coquel, S. Jin, J.-G. Liu, and L. Wang, *Well-posedness and singular limit of a semilinear hyperbolic relaxation system with a two-scale discontinuous relaxation rate*, To appear in Arch. Rat. Mech. Anal, (2014).
[6] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, vol. 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, third ed., 2010.
[7] A.-L. Dalibard, *Kinetic formulation for a parabolic conservation law. Application to homogenization*, SIAM J. Math. Anal., 39 (2007), pp. 891–915 (electronic).
[8] R. J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., 88 (1985), pp. 223–270.
[9] F. Golse, P. L. Lions, B. Perthame, and R. Sentis, *Regularity of the moments of the solution of a transport equation*, J. Funct. Anal., 76 (1988), pp. 110–125.
[10] L. Gosse and G. Toscani, *Space localization and well-balanced schemes for discrete kinetic models in diffusive regimes*, SIAM J. Numer. Anal., 41 (2003), pp. 641–658 (electronic).
[11] F. James, *Convergence results for some conservation laws with a reflux boundary condition and a relaxation term arising in chemical engineering*, SIAM J. Math. Anal., 29 (1998), pp. 1200–1223 (electronic).

[12] S. Jin and Z. P. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math., 48 (1995), pp. 235–276.

[13] M. A. Katsoulakis and A. E. Tzavaras, *Contractive relaxation systems and the scalar multidimensional conservation law*, Comm. Partial Differential Equations, 22 (1997), pp. 195–233.

[14] S. N. Kružkov, *First order quasilinear equations with several independent variables.*, Mat. Sb. (N.S.), 81 (123) (1970), pp. 228–255.

[15] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 5 (1978), pp. 489–507.

[16] R. Natalini, *Recent results on hyperbolic relaxation problems*, in Analysis of systems of conservation laws (Aachen, 1997), vol. 99 of Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 128–198.

[17] R. Natalini and A. Terracina, *Convergence of a relaxation approximation to a boundary value problem for conservation laws*, Comm. Partial Differential Equations, 26 (2001), pp. 1235–1252.

[18] B. Perthame, *Kinetic formulation of conservation laws*, vol. 21 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2002.

[19] M. Tournus, *An asymptotic study to explain the role of active transport in models with countercurrent exchangers*, SeMA Journal: Boletín de la Sociedad Española de Matemática Aplicada, 59 (2012), pp. 19–35.

[20] ———, *Modèles d'échanges ioniques dans le rein: théorie, analyse asymptotique et applications numériques*, PhD thesis, July 2013.

[21] M. Tournus, A. Edwards, N. Seguin, and B. Perthame, *Analysis of a simplified model of the urine concentration mechanism*, Network Heterogeneous Media, 7 (2012), pp. 989 – 1018.

[22] M. Tournus, N. Seguin, B. Perthame, S. R. Thomas, and A. Edwards, *A model of calcium transport along the rat nephron*, American Journal of Physiology - Renal Physiology, 305 (2013), pp. F979–F994.

[23] A. E. Tzavaras, *Relative entropy in hyperbolic relaxation*, Commun. Math. Sci., 3 (2005), pp. 119–132.

[24] S. Y. Zhang and Y. G. Wang, *Well-posedness and asymptotics for initial boundary value problems of linear relaxation systems in one space variable*, Z. Anal. Anwendungen, 23 (2004), pp. 607–630.