Well-posedness and qualitative properties of a kidney model

Magali Tournus, directed by Benoît Perthame, Aurelie Edwards, Nicolas Seguin

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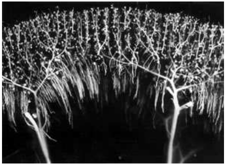

The kidney

Study of a simplified model

Well posedness and basic properties

The role of the pump

The kidney

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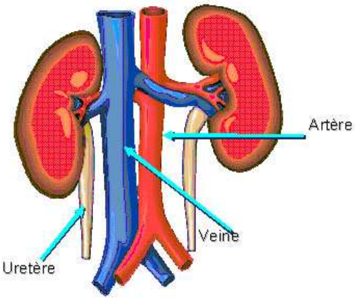
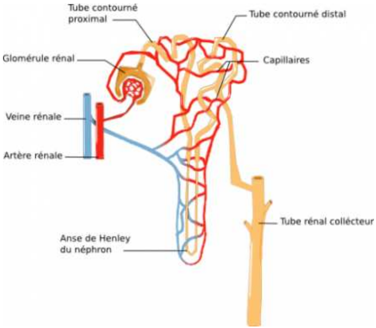


Figure:

The concentrating mechanism

Exchanges of solutes between the tubes makes the solute concentration variyng.



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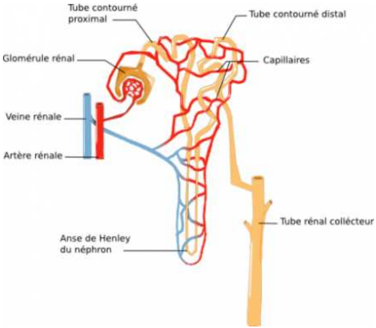

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Physiological attempts : At equilibrium, we expect a liquid with a high concetration "down below".

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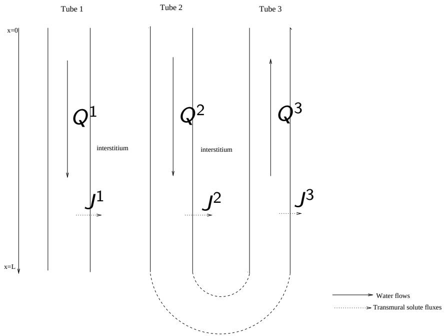

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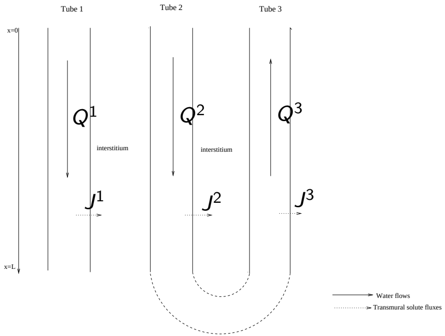

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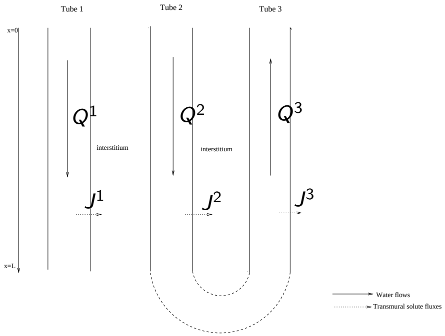

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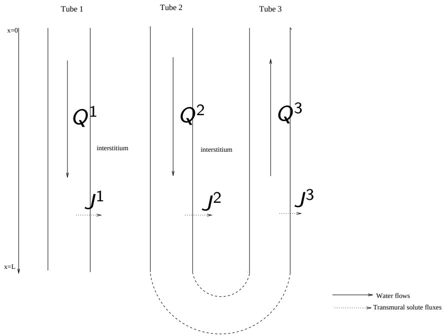

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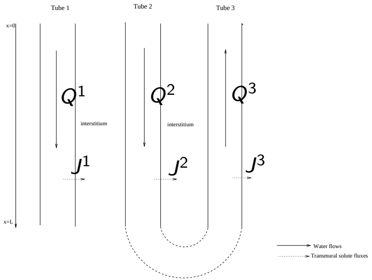
- A fluid circulates in the tubes
- C is concentration of a solute dissolved in the fluid (it can be any intensive quantity)

The simplified model



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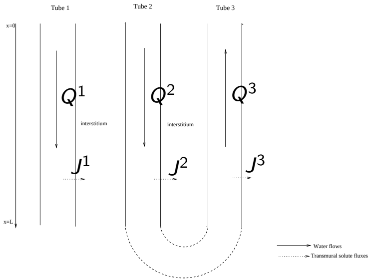

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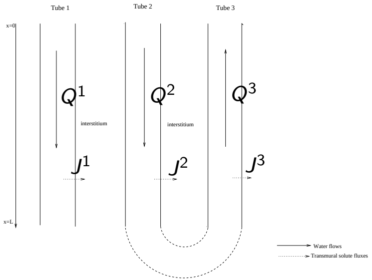

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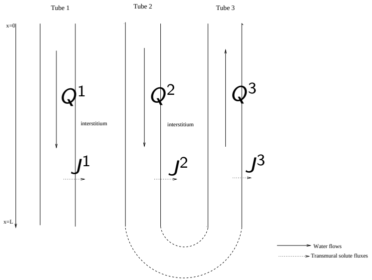

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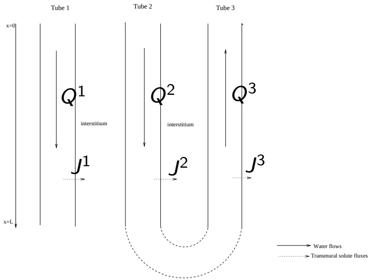

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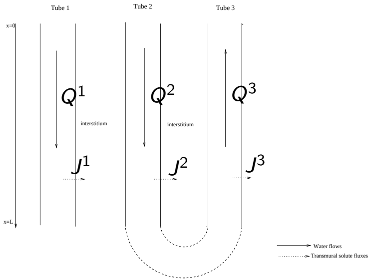

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  2. solute active transport in tube 3 : a pump extracts solute from tube 3 to the intersitium

The equations

$$\begin{cases} Q^1 \dfrac{dC^1(x)}{dx} = J^1(x), & x \in [0, L], \\ Q^2 \dfrac{dC^2(x)}{dx} = J^2(x), & x \in [0, L], \\ Q^3 \dfrac{dC^3(x)}{dx} = J^3(x), & x \in [0, L], \\ \quad C^1(0) = C_0^1, \qquad C^2(0) = C_0^2, & C^3(L) = C^2(L), \end{cases}$$

C_0^1 et C_0^2: Solute concetration at the inlet
Q^i: Water flow in tube i
J^i: Solute flux acroos wall tube i i.

Fluxes

$$J^1(x) = P^1(x)(C^{int}(x) - C^1(x)),$$
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P^i : Solute permeability of i
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- There is no solute accumlation in the interstitium
- This condition enables us to calculate C^{int}.

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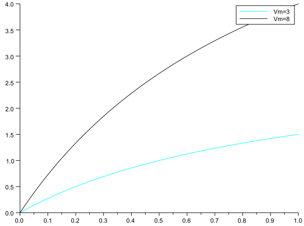

Figure:

The stationary and dynamic systems

The stationary system

$$\begin{cases} \dfrac{dC^1(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + F(C^3(x), x)\Big] - C^1(x), \\ \dfrac{dC^2(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + F(C^3(x), x)\Big] - C^2(x), \\ -\dfrac{dC^3(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + F(C^3(x), x)\Big] - C^3(x) - F(C^3(x), x), \\ C^1(0) = C_0^1, \qquad C^2(0) = C_0^2, \qquad C^3(L) = C^2(L). \end{cases} \tag{2}$$

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The dynamic system $t \geq 0$ et $x \in [0, L]$,

$$\begin{cases} \dfrac{\partial C^1}{\partial t} + \dfrac{\partial C^1}{\partial x} = \dfrac{1}{3}\Big[C^1 + C^2 + C_3 + F(C^3, x)\Big] - C^1, \\ \dfrac{\partial C^2}{\partial t} + \dfrac{\partial C^2}{\partial x} = \dfrac{1}{3}\Big[C^1 + C^2 + C^3 + F(C^3, x)\Big] - C^2, \\ \dfrac{\partial C^3}{\partial t} - \dfrac{\partial C^3}{\partial x} = \dfrac{1}{3}\Big[C^1 + C^2 + C^3 + F\big(C^3, x\big)\Big] - C^3 - F(C^3, x), \\ C^1(0,t) = C^1_0, \qquad C^2(0,t) = C^2_0, \qquad C^3(L,t) = C^2(L,t), \end{cases} \tag{3}$$

with initial conditions : $C^1(x, 0)$, $C^2(x, 0)$, $C^3(x, 0)$.

Main results

Theorem (Existence of a stationary solution)

Under assumptions (1), *there is a weak soolution to* (5), *and it is positive.*

Theorem (Existence and uniqueness of the dynamic problem solution)

There is a unique weak solution to the initial value problem (3), *which lies in* $BV([0, L] \times [0, T])$.

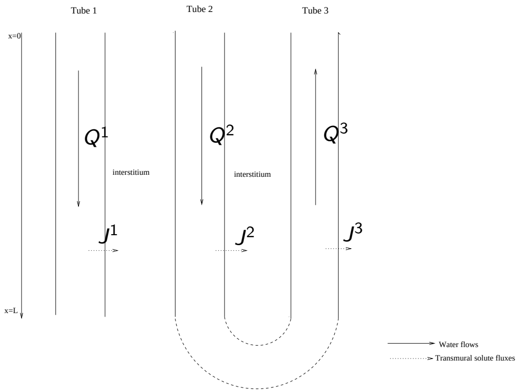
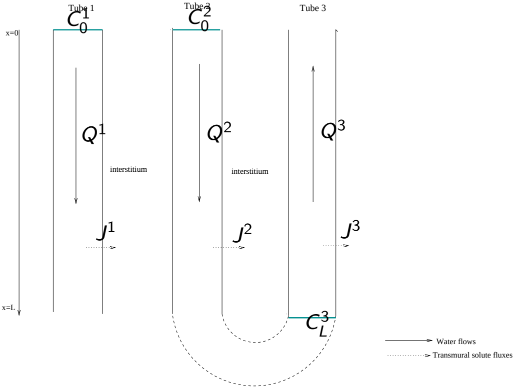


Figure:

Existence of a stationary solution

Sketch of the proof:



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- For all $\alpha > 0$, there exists a solution in $\Big(C[0,L]\Big)^3$ to the system

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- Compacity of the family $(C^1_\alpha, C^2_\alpha, C^3_\alpha) \Rightarrow$ After extraction, the sequence converges toward a solution to

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□

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- The limit function C is a weak solution (3).

Uniqueness of the dynamic problem solution

$$C^i(t=0) \geq 0, \qquad C^i(t=0) \in L^1(0,L), \qquad \frac{d}{dx}C^i(t=0) \in L^1(0,L). \quad (4)$$

Theorem (Contraction principle)

If $C(x,0)$ *et* $\widetilde{C}(x,0)$ *are two initial conditions, then the weak solutions satisfy*

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  We multiply each line by $\operatorname{sign}\left(C^i(x,t) - \widetilde{C^i}(x,t)\right)$
- We obtain

$$\begin{cases} \dfrac{\partial d^1}{\partial t} + \dfrac{\partial d^1}{\partial x} \le -\dfrac{2}{3}d^1 + \dfrac{1}{3}(d^2 + d^3 + G), \\ \dfrac{\partial d^2}{\partial t} + \dfrac{\partial d^2}{\partial x} \le -\dfrac{2}{3}d^2 + \dfrac{1}{3}(d^1 + d^3 + G), \\ \dfrac{\partial d^3}{\partial t} - \dfrac{\partial d^3}{\partial x} \le -\dfrac{2}{3}(d^3 + G) + \dfrac{1}{3}(d^2 + d^1). \end{cases}$$

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- Summing the line and integrating over $[0, L]$,

$$\frac{d}{dt}\int_0^L [d^1 + d^2 + d^3]dx \leq -d^1(L,t) - d^3(0,t) \leq 0,$$

□

Convergence of the dynamical solution toward stationary state

Theorem (Long time behavior and uniqueness of the stationary problem solution)

The solution C to the dynamic system converges to the unique solution $\overline{C}$ to the stationary system in L^1,

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Numerical method

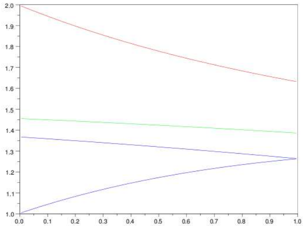
These results give us that

- There is a unique solution to the stationary system
- We can efficietly approach this solution by solving the dynamic problem and let the time evolve

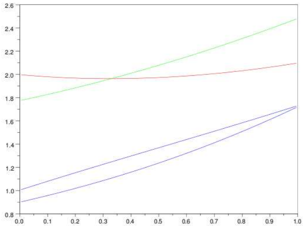
We use a finite-volume type scheme.
The CFL condition :

$$\Delta t \leq \frac{3\Delta x}{3 + 2\Delta x + 2\Delta x V}.$$

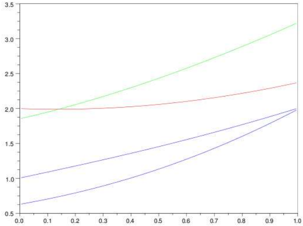
Some results



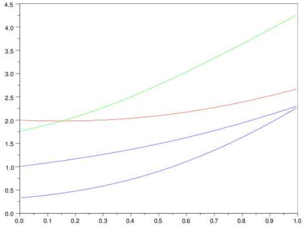
(Vm=0)



(Vm=3)



(Vm=5)



(Vm=8)

The role of the pump

- The concentration increases in all tubes along the axis of x
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We want to identify the profile of C when $V \longrightarrow \infty$.

$$\begin{cases} \dfrac{dC^1(x)}{dx} = \dfrac{1}{3}\left[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1 + C^3(x)}\right] - C^1(x), \\ \dfrac{dC^2(x)}{dx} = \dfrac{1}{3}\left[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1 + C^3(x)}\right] - C^2(x), \\ -\dfrac{dC^3(x)}{dx} = \dfrac{1}{3}\left[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1 + C^3(x)}\right] - C^3(x) - V\dfrac{C^3(x)}{1 + C^3(x)}, \\ C^1(0) = C^1_0, \qquad C^2(0) = C^2_0, \qquad C^3(L) = C^2(L). \end{cases} \tag{5}$$

Purpose: Explain the behavior of C_V when $V \longrightarrow \infty$

Theorem (Asymptotics)

Solutions to (5) *satisfy*

$$(C_V^1, C_V^2, C_V^3) \underset{V \longrightarrow +\infty}{\longrightarrow} (C^1, C^2, C^3) \qquad L^p(1 \leq p < \infty), \ a.e.,$$

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with

$$\begin{cases} C^1(x) = \dfrac{C_0^1 + C_0^2}{2} + \dfrac{C_0^1 - C_0^2}{2} e^{-x}, \\ C^2(x) = \dfrac{C_0^1 + C_0^2}{2} + \dfrac{C_0^2 - C_0^1}{2} e^{-x}, \\ C^3(x) = 0. \end{cases} \quad \begin{cases} \mu^1 = \dfrac{1}{2}\Big[(C_0^2 - C_0^1)e^{-x} + B\delta_L\Big], \\ \mu^2 = \dfrac{1}{2}\Big[(C_0^1 - C_0^2)e^{-x} + B\delta_L\Big], \\ \mu^3 = B\delta_L. \end{cases}$$

where

$$B = \lim_{V \to \infty} C_V^3(L).$$

A priori bounds and compact embeddings

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with

$$\lim_{V\to\infty} C_V^3(L) = B \geq 0, \qquad \lim_{V\to\infty} C_V^3(0) = A \geq 0. \tag{6}$$

Then, we prove $A = 0$, report in the system and do "analytical computations"

The boundary layer

To perform numerical simulations, we have to precise the shape of the boundary layer.

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Theorem (The boundary layer)

The limits of the boundary values are

$$C_V^1(L) \underset{V \longrightarrow +\infty}{\longrightarrow} C_0^1 + C_0^2$$

$$C_V^2(L) = C_V^3(L) \underset{V \longrightarrow +\infty}{\longrightarrow} C_0^1 + C_0^2 + (C_0^2 - C_0^1)e^{-L}$$

The behavior of C_V^3 for $x \simeq L$ is given by the inequalities

$$C_V^3(x) \leq C_V^3(L) \exp\Big(- \frac{2}{3} V M(L-x)\Big) + \frac{K}{V}\Big[1 - \exp\Big(- \frac{2}{3} V M(L-x)\Big)\Big]$$

$$C_V^3(x) \geq C_V^3(L) \exp\Big(- \frac{2}{3} V (L-x)\Big) + \frac{\overline{K}}{V}\Big[1 - \exp\Big(- \frac{2}{3} V (L-x)\Big)\Big]$$

Numerical illustrations

The difficulties are : When V grows,

Magali Tournus, directed by Benoît Perthame, Aurelie Edwards, Nicolas Seguin

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Numerical illustrations

The difficulties are : When V grows,

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- The number of mesh necessary to see the phenomenon becomes high next to L

Numerical illustrations

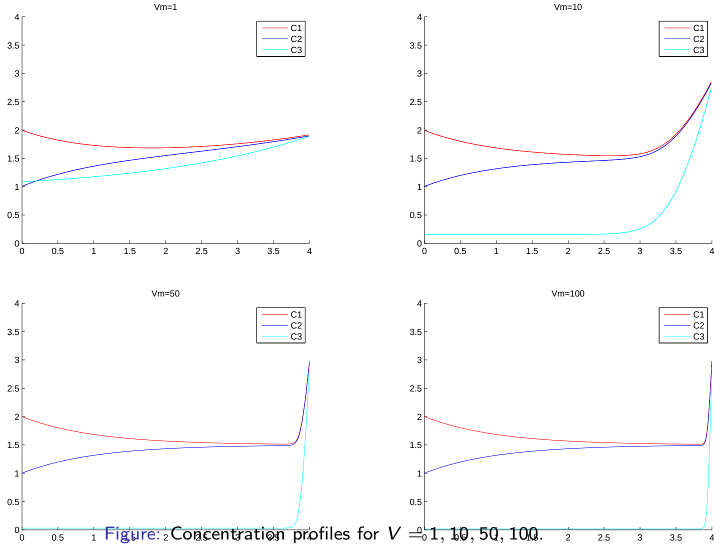
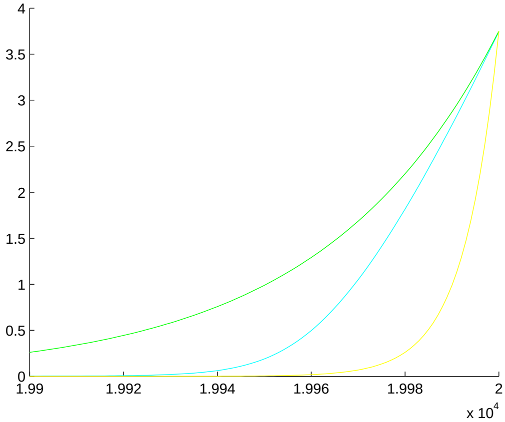


Figure: Concentration profiles for $V = 1, 10, 50, 100$.



4
3.5
3
2.5
2
1.5
1
0.5
0
1.99
1.992
1.994
1.996
1.998
2
x 10^4

Perspectives and ongoing projects

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